Geophysical Fluid Dynamics: a survival kit

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Some questions every graduate student in atmosphere/ocean science should be able to answer:

> *If I change the rotation rate of Earth, what happens to the circulation? Start with Hadley cell; what is a relevant conserved quantity for air parcels uplifted by a Hadley cell? How does that relate to winds away from the equator? How does that related to winds in mid-latitude & in part, to meteorological activity there? Why is there a Hadley cell? If an air parcel in the northern hemisphere is heading directly east, which way does the Coriolis effect deflect the parcel? what is geostrophic balance? where does it apply? a) derive the thermal wind balance b) derive the Rossby radius of deformation by estimating the distance a gravity wave travels in one rotational period for a two layer stratified fluid c) explain the relationship between interface displacements of a two layer stratified fluid and the surface elevation. What are the orders of magnitude of both parameters accross the Gulf Stream and along the Equatorial Pacific ?* a) derive an equation for the mixed layer depths evolution in a constantly stratified fluid (N2(z)=const) subject to a constant in time surface buoyancy flux (B0(t)=const) 2. buoyancy forced mixed layers a) derive an equation for the mixed layer depths evolution in a constantly stratified fluid (N2(z)=const) subject to a constant in time surface buoyancy flux (B0(t)=const) b) what is the relatinship between surface heat flux and surface buoyancy flux. By how much does it vary between the tropics (31C) and the marginal ice zone (-2C) c) discuss under which circumstance an oceanic mixed layer might not obey the idealized equation 2a) (think about other forcings and the equation of state...) 3. potential vorticity a) derive the potential vorticity equation for a beta plane including lateral mixing and wind forcing b) use the equation to explain the dynamical balances in the i)subtropical and ii) subpolar gyre. Why are the boundary currents in the western part of the basin c) use the equation to explain the circulation pattern in a one hemisphere sector ocean with a mass source in the north and constant upwelling elsewhere. In which way does this simple scenario relate to the deep water circulation? where are potential problems?

1 The governing equations

1.1 Conservation equations

Using the conservation of mass, enthalpy and momentum[1], we obtain the **equations of motion** :

$$\rho \frac{D_a \boldsymbol{u_a}}{Dt} = -\boldsymbol{\nabla} p + \rho \boldsymbol{g} + \mathfrak{F} \qquad \text{momentum} \tag{1.1a}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = 0 \qquad \text{mass} \tag{1.1b}$$

$$C_p \frac{dT}{dt} - \frac{T}{\rho} \alpha \frac{dp}{dt} = \kappa \nabla^2 T + \dot{Q} \qquad \text{enthalpy} \tag{1.1c}$$

here α is the thermal expansion coefficient and $\dot{Q}$ the heating rate, including the effect of radiation, latent-heat release, geothermal heating, etc... The energy equation is just the first law of thermodynamics ; it takes quite different forms for the atmosphere and the ocean. We would write similar equations for every other quantity we wish to consider : in the ocean, salinity S is a good one ; in the atmosphere, specific humidity (q) is quite essential too, but in any event we should add one conservation equation per chemical species we consider.
The momentum equation is here written in an "absolute frame of reference" (i.e. fixed with respect to stars), denoted by the subscript "a". Since it is more practical to look at it from a fixed point on Earth, we must rotate with the fluid. In addition, the Earth has a spherical shape, and we like to use this geometry, which changes the description a bit more. Namely :

1.2 Spinning your head

Written in our rotating frame of reference (rotation is described by the vector $\boldsymbol{\Omega}$), the momentum equations become :

$$\rho \left[\frac{d\boldsymbol{u}}{dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u} \right] = -\boldsymbol{\nabla} p + \rho \boldsymbol{g} + \mathfrak{F} \tag{1.2}$$

Now $\boldsymbol{u}$ is the velocity in our Earthy reference frame and we write the Lagragian derivative by the usual $\frac{d}{dt}$. The change of referential has given birth to 2 new forces : the Coriolis force and the centrifugal force. The latter can be quickly swallowed by the gravity term $\boldsymbol{g} = \boldsymbol{g}^* - \Omega^2 \boldsymbol{R}$ (gravitational force = gravity - centrifugal force) since it is such a minor correction on Earth. However, the big news is that we now have to deal with a Coriolis force, which will clearly appear as one of the most important in the system. It is only an illusion of our frame of reference, though.

1.3 A well-rounded planet

Now, if we write that in spherical coordinates (r, λ, ϕ) (see *Holton* (1992) p33-38), we get in addition some **curvature terms** that stem from the Lagrangian derivatives of (rotating) unit vectors (e.g. $v\frac{d\boldsymbol{j}}{dt}$). They are usually neglected, since most of the time we look at scales $L \ll a$, but we should keep in mind they must be retained on a planetary scale (e.g. GCM).

[1]These quantities are not always strictly conserved, as Dick Ou pointed out in the case of momentum, but at least we can formulate a statement relating their rate of change to sources and sinks. The best derivation I have seen is in the first chapter of Marc Spiegelman's MMM notes.

Now, for the sake of it, let's write the complete equation for v, say :

$$\frac{dv}{dt} + \frac{u^2 \tan\phi}{r} + \frac{uw}{r} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mathfrak{F}_y}{\rho} \tag{1.3}$$

($f = 2\Omega \sin\phi$ is the Coriolis parameter). From then on we will try to simplify it as much as we can.

1.4 What to do with friction ?

Classically, in the interior, we write the frictional force per unit mass:

$$\frac{1}{\rho}\mathfrak{F} = \boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \nu\nabla^2\boldsymbol{u} \tag{1.4}$$

with ν the kinematic viscosity ($\mathrm{m^2s^{-1}}$), because we think that mixing by small-scale (turbulent) processes is best described by this analogy to molecular processes. This is only to hide our abysmal ignorance of turbulence. Because it is so annoying, we try to ignore it outside boundary-layers, but the formulation of turbulent processes is clearly the weakest link in our edifice....
Typical values of momentum diffusivity depend on the scales considered... Eddy momentum transfer corresponds roughly to $\nu \sim 2000\ \mathrm{m^2s^{-s}}$, but in a GCM it is resolution-dependent because most numerical schemes cause negative diffusion.
NB : for the air/sea momentum exchange, we like to write, in the upper layer of the ocean :

$$\frac{1}{\rho}\mathfrak{F} = \frac{\partial}{\partial z}\boldsymbol{\tau_s} \tag{1.5}$$

where τ_s is the wind-stress. When we integrate this over the depth of mixed-layer, this is effectively like considering turbulent momentum fluxes as a **body force**.

1.5 The thermodynamic energy equation

To make progress on this one, we need an **equation of state**, relating ρ, T and p.

1.5.1 In the Atmosphere

Up in the air it is the ideal gas law $P = \rho RT$ (but recall that R is gas-specific gas "constant" : it is normalized by the mean molecular weight. Read Salby, p5). We like to consider as our primary thermodynamical variable the potential temperature :

$$\theta = T\left(\frac{p_\circ}{p}\right)^{R/C_p} \tag{1.6}$$

which is cute because in the absence of diabatic heating (typically $-L\frac{dq}{dt}$), it relates to specific entropy s by :

$$d\mathrm{s} = C_p\, d\ln\theta \tag{1.7}$$

(*Holton* (1992), p 51,52). By construction, it is conserved during adiabatic motion (i.e. isentropic displacement, since we assume implicitly that they are reversible). Hence, we write the first law in terms of potential temperature :

$$\frac{1}{\theta}\frac{d\theta}{dt} = \frac{1}{C_pT}\left[\kappa\nabla^2 T + \dot{Q}\right] \tag{1.8}$$

1.5.2 In the ocean

Using the near-incompressibility of the ocean (cf (2.3)) , the energy equation becomes an advection-diffusion equation for temperature. We can write a similar conservation equation for the salt. The real equation of state of seawater is terribly non-linear, but locally one can linearize it :

$$\rho = \rho_\circ \left[1 + \alpha(T - T_0) + \beta(S - S_0)\right] \tag{1.9}$$

here $\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial S}\right)_T \quad \alpha = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_S$. The most important non-linearities are (1) that α increases with T (the same ΔT produces a bigger decrease in density at 20 degrees, than at 0 degrees, say). Also, seawater is more compressible (i.e. sensitive to pressure variations) at low temperatures, leading to thermobaric instability (2) that β increases at low temperature (cold water density is more sensitive to salinity variations). All of these combine of course, so the seawater physics in polar region is critical to deep-water formation (cabelling, thermobaric effect, not to mention brine rejection) For more details, see the "Equation of state" review by Steve Gaurin.

Because the advection-diffusion operators are linear, and assuming that $\kappa_T = \kappa_S = \kappa$, we can translate :

$$\frac{dT}{dt} = \kappa_T \nabla^2 T + \frac{Q_T}{\rho_\circ C_p} \tag{1.10}$$

$$\frac{dS}{dt} = \kappa_S \nabla^2 S + F_S \tag{1.11}$$

into :

$$\frac{d\rho}{dt} = \kappa \nabla^2 \rho + \mathfrak{B} \tag{1.12}$$

where $\mathfrak{B}$ is a buoyancy flux, typically zero in the interior but critically important near the surface. [2]

NB :

- although the energy equation contains a Laplacian operator in ρ, mass does not diffuse *per se* (it is just a convenient writing).
- That $\frac{d\rho}{dt}$ is not zero does not compromise the incompressibility (cf below) of seawater. This is really a statement of heat and salt conservation, not compressibility.
- Whenever $\kappa_T \neq \kappa_S$, a process called **double-diffusion** can occur, leading to some "salt fingers". That looks very very cool, but it happens in such rare circumstances that it's quite unclear how much global ocean mixing is due to that. Wins you points with GFD geeks, though.
- As an aside, another important consequence of $\kappa_T = \kappa_S$ is that in a T-S diagram, water masses mix along straight lines. If $\kappa_T \neq \kappa_S$, they mix along a hyperbola.

[2] More precisely :

$$\mathfrak{B}(z = 0) = -\frac{\alpha}{C_p} Q_s + \frac{\beta S}{1 - S}(E - P) \tag{1.13}$$

where Q_s is the net surface heat flux and $E - P$ the net surface freshwater flux

- Usually vertical diffusion dominates, since gradients are so strong in the vertical direction. The vertical mixing coefficient, or turbulent diffusivity, often written K_z, is about $10^{-5}\ \mathrm{m^2s^{-1}}$ in the interior (weak, but still 2 orders of magnitude greater than molecular diffusivity), and $10^{-4}\ \mathrm{m^2s^{-1}}$ or more over rough topography (continental slopes, ridges, seamounts, etc). More rigorously, we should be speaking of diapycnal diffusivity (which only coincides with vertical diffusivity when isopycnals are horizontal, which is certainly not the case in the Southern Ocean –hence it is the locus of major eddy-driven diapycnal fluxes.
- Comparing (1.8) and (1.12) emphasizes the analogy between θ in the atmosphere and ρ in the ocean.

Conclusion : we have pushed mathematical transformations[3] of our basic conservation equations (which are only the mathematical expression of physical principles) as far as we could. This would be enough if we were able to resolve all scales of motion, but unfortunately that is not a luxury that these non-linear partial differential equations afford us. Instead, the usual procedure (since J.G. Charney) is to evaluate the relative magnitude of the different terms involved in our equations, to try and simplify them a bit.

2 Scale Analysis

The idea of scale analysis is to look at what the real world has to say *before* jumping in the resolution of non-linear PDEs, so as to retain only the most physically relevant terms for each problem. It is very problem-dependent, but for historical reasons, this was first done in midlatitudes.

2.1 Heuristic scaling

In Charney's words (1948) :

> *[...] This extreme generality whereby the equations of motion apply to the entire spectrum of motions - to sound waves as well as cyclone waves - constitutes a serious defect of the equations from the meteorological point of view. It means that the investigator must take into account modifications to the large-scale motions of the atmosphere which are of little meteorological importance and which only serve to make the integration of the equations a virtual impossibility.*
> J.G. Charney

The basic idea is to replace each variable by its characteristic scale (u by U, (x, y) by L, etc..) and evaluate the gross magnitude of the terms. It is done for example in *Cushman-Roisin* (1994) p41. The goal is to focus on a scale of interest (synoptic, mesoscale, hair-length ?) and filter out the motion on irrelevant scales.

2.1.1 Scale analysis of momentum equation

not done here.

[3] I must admit I cheated a little bit for the energy equation in the ocean, by anticipating the incompressibility, which will soon we demonstrated

2.1.2 Scale analysis of continuity equation

1. **The Boussinesq approximation**
   This is a ubiquitous approximation, yet rarely rigorously justified. In words, it states that **density can be considered a constant except in the terms where it is multiplied by g**. One first way to obtain it is to split the density field into 2 parts: ρ_s is the mean stratification and a perturbation ρ'. Then write the continuity equation :

$$\frac{1}{\rho_s}\left(\frac{\partial \rho'}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho'\right) + \frac{w}{\rho_s}\frac{d\rho_s}{dz} + \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \tag{2.1}$$

With a midlatitude synoptic scaling approach, you can show that only the last 2 terms matter in the atmosphere (*Holton* (1992), p46).

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} + w\frac{d}{dz}\log\rho_s = 0 \tag{2.2}$$

In the ocean, it is even more dramatic, and only the last term survives the treatment :

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \tag{2.3}$$

(this says that volume is conserved, since density is almost constant). It is strictly valid only for an incompressible fluid, but it is perhaps the most solid approximation we make about the ocean.

However, the problem of scale analysis is that it is scale-dependent by nature. Here's a derivation I like better, since it is more general.

Split the density and pressure field into 2 parts :

$$\rho = \rho_s(z) + \rho'(x,y,z,t) \qquad \rho' \ll \rho_s \tag{2.4a}$$
$$P = P_s(z) + p(x,y,z,t) \qquad p' \ll P_s \tag{2.4b}$$

ρ_s is the mean stratification, considered constant. It is in **hydrostatic equilibrium** with the mean pressure $P_s(z)$, in other words, P would reduce to P_s if the fluid were at rest. Defined this way, ρ' and p' are the density and pressure field associated with the *motion*. Such a linearization is made possible by the strong **stratification** of the ocean.

Now, in the zonal and meridional momentum equations, dividing by ρ is not much different from dividing by ρ_s (or even a constant $\rho_\circ = 1025\ \mathrm{kgm}^{-3}$), but such is not the case in the vertical momentum balance :

$$\rho\frac{dw}{dt} = -\frac{\partial P}{\partial z} - \rho g + \mathrm{junk} \tag{2.5}$$

substituting (2.4) into the latter, the term $\frac{\partial P_s}{\partial z} - \rho_s g$ vanishes by construction, and the remaining equation is, after dividing by $\rho_\circ \simeq \rho_s$:

$$\frac{dw}{dt} = -\frac{1}{\rho_\circ}\frac{\partial p}{\partial z} - \frac{\rho'}{\rho_\circ}g \tag{2.6}$$

In the horizontal momentum equations, the pressure gradient force becomes :

$$-\frac{1}{\rho}\boldsymbol{\nabla} P \simeq -\frac{1}{\rho_\circ}\boldsymbol{\nabla} p' \tag{2.7}$$

which says that the only terms driving the motion are the perturbation terms. For this reason, in all GFD textbooks , ρ' is replaced by ρ after a blink of an eye, and the "pressure" p is in fact the *dynamic pressure* $p' = P - P_s$. As far as I know, non-Boussinesq effects are neglected in large-scale ocean dynamics, but are critical in convective parameterizations. The term $-\frac{\rho'}{\rho_\circ}g$ is called the **buoyancy**.

In the atmosphere, density variations are a lot harder to ignore : for instance, in the hydrostatic approximation, and for an isothermal atmosphere, it should decrease exponentially with height.... For this reason we kept $w\frac{d}{dz}\log\rho_s$ in (2.2). BUT, there is a magical trick to make the mass conservation equation look like $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$: **pressure coordinates** (*Holton* (1992) p21-23). However, the Boussinesq approximation is sometimes used in wave problems, or in the planetary boundary layer (well-mixed $\Rightarrow$ constant density). The buoyancy writes $g\frac{\theta}{\theta_\circ}$ in the atmosphere.

NB for atm and ocean : this approximation filters out soundwaves (i.e. if you are asked about acoustic tomography in the ocean, don't use it !)

2.2 A more rigorous scaling : Non-dimensionalization and asymptotic expansions

A more rigorous approach is to non-dimensionalize the equations, which, at first, is somewhat independent of the situation (you don't have to specify the scales a priori, as done before). However, the way to relate time and space scales, for example, is quite subjective. In the following example I consider an advective timescale. :

$$(u, v) = U\,(u^*, v^*) \tag{2.8a}$$

$$(x, y) = L\,(x^*, y^*) \tag{2.8b}$$

$$t = \frac{L}{U}t^* \tag{2.8c}$$

$$\boldsymbol{\nabla} P = \frac{\Delta P}{L}\boldsymbol{\nabla}^* P^* \tag{2.8d}$$

$$f = f_\circ f^* \tag{2.8e}$$

and so on and so forth... The stars denote non-dimensional variables (between 0 and 1, usually). Then, the y-momentum equation (1.3), for instance, becomes :

$$\frac{U^2}{L}\frac{dv^*}{dt^*} + \frac{U^2}{r}(u^*)^2\tan(\phi) + f_\circ U\, f^* u^* = -\frac{\Delta P}{\rho_\circ L}\frac{\partial P^*}{\partial y^*} + \nu\frac{U}{L^2}\nabla^{*2}v^* \tag{2.9}$$

Dividing by $f_\circ U$, and scaling ΔP as $\rho_\circ g H$, we get (after some algebra) :

$$\epsilon\frac{dv^*}{dt^*} + \epsilon\frac{L}{r}(u^*)^2\tan(\phi) + f^* u^* = -\epsilon^{-1}\frac{gH}{f_\circ^2 L^2}\frac{\partial P^*}{\partial y^*} + \mathrm{Ek}\frac{\partial^2 v^*}{\partial z^{*2}} \tag{2.10}$$

where $\epsilon = \frac{U}{f_\circ L}$is the Rossby number, and $\mathrm{Ek} = \frac{\nu}{f_\circ H^2}$ is the Ekman number. The ratio $\frac{gH}{f_\circ^2 L^2} = \left(\frac{L_D}{L}\right)^2 = \mathrm{Bu}$ is the Burger number. Then, it is possible to explore the parameter range and get simplified equations for each context. When $\mathrm{Ek} \ll \epsilon \ll 1$ and $\mathrm{Bu} \sim 1$, which implies $L \ll r$, we get geostrophic equilibrium, for example. It is customary to use asymptotic expansions in ϵ, considered a small parameter :

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \tag{2.11}$$

Of course, the same transformations are applied to the other equations (u,ρ, etc.), in which all variables are expanded in this way. Then the idea is to plug this asymptotic series into the equations and gather all the powers of ϵ:

- To zeroth order, you get geostrophic equilibrium
- To first order, you get quasi-geostrophic theory
- To higher orders you get peculiar beasts like the Korteweg de Vries (KdV) equation (3rd order), solitons, etc..

This method is called **singular perturbation theory** and was initially developed in celestial mechanics[4]. I think it is the only rigorous way to justify all the dynamical approximations. In our case, it states that the velocity field is **almost geostrophic** except for small perturbations on the order of a power of the Rossby number (assumed small, which can only be true far away from the equator, for large-scales and modest velocities). The first of these corrections is the quasi-geostrophic wind (or current), and it is actually where most of the interesting dynamics happen (see section 5). Read *Pedlosky* (1987) (yes, the whole opus) for further details on the method.
For a compact review of important non-dimensional number used in fluid dynamics (and GFD in particular), please ask for the table by Michael Allison (which I was too lazy to write down in LaTeX).

2.3 Classical approximations : geometrical simplifications

2.3.1 Sphericity

The first usual approximation is that departures of the Earth's surface from sphericity are negligible : $r(\lambda, \phi) = a \simeq 6370$ km

2.3.2 The thin-shell approximation

The depth of the atmosphere (30 km) is much smaller than the earth radius (6370 km). Therefore, it is customary in meteorology to assume that the atmosphere is a thin shell ($z \ll a$ and $r \simeq a$). However, if this hypothesis is retained without care, the principle of conservation of angular momentum may be violated (Phillips, 1966). It is necessary to make the additional hypothesis that the horizontal component of the earth rotation is negligible. The immediate practical consequence is that the gravitational force is considered to have the same value $g_\circ \simeq 9.81\ \mathrm{ms}^{-2}$ throughout the volume of the ocean and the atmosphere. See *Holton* (1992), p8 and *Cushman-Roisin* (1994) (p19) for a discussion.

[4] An example is found on http://www.math.arizona.edu/ lega/583/Spring99/lectnotes/PT2.html

2.3.3 f-plane / β-plane approximations

The Coriolis parameter varies as the sine of latitude : $f = 2\Omega \sin(\phi)$. This expression can be expanded in Taylor series, around a central latitude ϕ_0 and longitude λ_0 :

$$y = a(\phi - \phi_0) \tag{2.12}$$

$$x = a\cos(\phi_\circ)[\lambda - \lambda_\circ] \tag{2.13}$$

1. **The β-plane** For small meridional displacements, we may retain only the first 2 terms of this expansion of the sine :

   $$f = f_\circ + \beta y \tag{2.14}$$

   where $\beta = \left(\frac{\partial f}{\partial y}\right)_{\phi_0} = \frac{2\Omega}{a}\cos\phi_\circ$. The approximation is only valid for $\frac{\beta L}{f_\circ} \ll 1$ in midlatitudes, where common values are $f \sim 8 \times 10^{-5}\ \mathrm{s}^{-1}$ and $\beta \sim 2 \times 10^{-11}\ \mathrm{m}^{-1}\mathrm{s}^{-1}$. ($f$ is typically one order of magntiude smaller in the Tropics)

   Conceptually, this is equivalent to neglecting locally the curvature of the Earth, as if we were lying on a plane tangent to the Earth in one point (x_0, y_0). f is replaced by $f_\circ$ everywhere, except when it is differentiated, in which case it is replaced by β. The approximation only retains the effect of Earth's curvature on the meridional variations of the Coriolis parameter, while discarding all other effects of the curvature. *Pedlosky* (1987) shows nicely that this is what really matters for geophysical fluids (the quasi-geostrophic potential vorticity equation ends up having the same *exact* form on a sphere or a beta-plane). The advantage is that it makes the equations considerably easier to solve (constant coefficients...). In meteorology, the geopotential then becomes a streamfunction – so does the pressure in a Boussinesq ocean – which reduced a vector field to a scalar function, always a good thing.

   This approximation is a building block of quasi-geostrophic theory, as we shall see in Section5. It is also exceedingly used near the equator ($f_\circ = 0$), where the model is referred to as the "equatorial β-plane" : $f = \beta y$. It is actually pretty accurate, and is very useful in understanding the qualitative behavior of equatorial waves.

   NB : The worst part of the approximation is not that $\beta = cst$: it is that meridians do not converge towards the pole (a geometric, not algebraic, drawback).

2. **The f-plane** For very small displacements, we may only retain $f = f_\circ$: this is the **f-plane**. It is typically used in midlatitudes where f is large, and for scales that do not feel the curvature of the Earth (ex : gravity waves). It is also very useful to understand experimental results in rotating tanks. The closest thing on Earth to an f-plane is the Arctic ocean ($\beta = 0$ at the Pole).

2.4 The primitive equations

After having made these essential simplifications (Boussinesq, incompressibility, hydrostatism, etc..), the equations of motion take a more tractable form (*Cushman-Roisin* (1994) p43) :

$$(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\, u - fv = -\frac{1}{\rho_\circ}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \qquad \text{x-momentum} \tag{2.15a}$$

$$(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\, v + fu = -\frac{1}{\rho_\circ}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 u}{\partial z^2} \qquad \text{y-momentum} \tag{2.15b}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \qquad \text{z-momentum} \tag{2.15c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{continuity} \tag{2.15d}$$

$$(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\, \rho = \kappa \frac{\partial^2 \rho}{\partial z^2} \qquad \text{density} \tag{2.15e}$$

5 equations, 5 variables (u, v, w, p and ρ). This is the system that most GCMs solve. David Rind likes to point out that this is only one type of filter ("pink glasses"), and that it is equally valid to view the system in terms of vorticity and divergence, deformation.

2.5 Why they prefer mesoscales : the Rossby radius of deformation

Here comes the holy grail of GFD, the intrinsic length scale of geophysical fluids : the radius of deformation, of which we generously give several definitions (all equivalent).

1. **The wave viewpoint**:

   L_D is the lateral scale travelled by a gravity or Kelvin wave in one inertial period ($T = f^{-1}$), i.e.

   $$L_D = \frac{c}{f} = \begin{cases} \frac{\sqrt{gH}}{f} & \text{External Rossby radius,} \\ \frac{NH}{f} & \text{Internal Rossby radius.} \end{cases} \tag{2.16}$$

   depending barotropic or baroclinic waves are considered. The internal Rossby radius is usually smaller than the external one by at least one order of magnitude (reflecting the factor 100 between barotropic and baroclinic wave speed) .

2. **The non-dimensional number viewpoint : rotation vs. stratification**
   L_D can be defined as the length scale at which rotation and stratification are equally important, hence the Froude number and the Rossby number should be both of order unity : $\mathrm{Fr} \sim \mathrm{Ro} \sim 1$, or :

   $$\frac{U}{NH} \sim \frac{U}{fL} \Rightarrow L \sim \frac{NH}{f} \tag{2.17}$$

   Important subtlety : H must be the vertical scale of the *motion*, which not the same that the depth of the ocean, for example. Sometimes it is the scale height, but it can also depend on the vertical velocity shear, for example. Bottom-line: it must be carefully defined.

3. **The instability viewpoint**: the first baroclinic Rossby radius it is the preferential scale at which baroclinic instability develops (cf section 6.1)

4. **The practical viewpoint :** In a shallow-water model, it is the length scale that combines all three fundamental constants of the problem : Ω, g and H.

3 Prevailing balances

3.1 Gradient-Wind balance

Building on the results of the scale analysis (section 2), we only retain the largest terms in each equation (or, more appropriately, the one with the same order in Rossby number expansion). For a balance between lateral advection, Coriolis force and pressure gradient force, it is convenient to define local curvilnear coordinates $(\vec{e}_r, \vec{e}_n)$. $\vec{e}_r$ follows the motion, and $\vec{e}_n$ is at a right angle with it, and the local **radius of curvature** R is positive to the left of the trajectory, by convention. Then the momentum balance becomes, along $\vec{e}_n$:

$$\frac{V^2}{R} + fV = -\frac{1}{\rho}\frac{\partial p}{\partial n} \tag{3.1}$$

NB : curvature terms like $\frac{uv}{a}\tan\phi$ (cf section 1) are not the same as $\frac{V^2}{R}$ written here. Do not confuse the curvature of the Earth and the local radius of curvature of a trajectory in the "natural" coordinate system (sphericity of Earth vs non-linear advection around, say, a hurricane).
From this equation stem 3 balances, depending on which term can be neglected in front of the other two: geostrophy, cyclostrophy and inertial motion (*Holton* (1992), p61-69). The most important is:

3.1.1 Geostrophy

To zeroth order in Ro, the horizontal momentum balance is the most famous set of equations in GFD, geostrophic equilibrium:

Ocean	Atmosphere	
$-fv = -\frac{1}{\rho_\circ}\frac{\partial p}{\partial x}$	$-fv = -\frac{\partial \Phi}{\partial x}$	(3.2)
$+fu = -\frac{1}{\rho_\circ}\frac{\partial p}{\partial y}$	$+fu = -\frac{\partial \Phi}{\partial y}$	(3.3)

This balance is omnipresent away from the equator. The reason for this is that it actually represents a minimum of energy for a geophysical system (*Cushman-Roisin* (1994) demonstrates this nicely in p 187,189).

3.1.2 Cyclostrophic equilibrium

Explains why you can have very strong winds around low-pressure systems and not around high-pressure systems.

3.2 The thermal wind balance (TWB)

1. **Keep it stupid simple**
   Once we have the previous balances, TWB follows trivially by differentiating (3.3) and (3.2) with respect to the vertical coordinate. For u :

$$f\frac{\partial u}{\partial z} = -\frac{1}{\rho_\circ}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right) \tag{3.4}$$

   then we invert the order of differentiation[5] : $\partial_{z\alpha}p = \partial_{\alpha z}p$, where α is either x or y. If we now use the hydrostatic equilibrium $\partial_z p = -\rho g$, we obtain :

$$f\frac{\partial u}{\partial z} = +\frac{g}{\rho_\circ}\frac{\partial \rho}{\partial y} \tag{3.5}$$

$$f\frac{\partial v}{\partial z} = -\frac{g}{\rho_\circ}\frac{\partial \rho}{\partial x} \tag{3.6}$$

   The same is true for the atmosphere (using this time $\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$ as the hydrostatic relation) :

$$p\frac{\partial u}{\partial p} = +\frac{R}{f}\frac{\partial T}{\partial y} \tag{3.7}$$

$$p\frac{\partial v}{\partial p} = -\frac{R}{f}\frac{\partial T}{\partial x} \tag{3.8}$$

2. **Add a little subtlety** A more subtle way of deriving the TWB uses vorticity conservation (*Pedlosky* (1987), p42). In the absence of friction, we can write :

$$(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\omega}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1}{\rho^2}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p \tag{3.9}$$

   Now, except from the time-derivative, we can approximate the total vorticity by its planetary component $\boldsymbol{\omega} \simeq 2\boldsymbol{\Omega}$ (this works for a small Rossby number), and we can also neglect the time-derivative. This yields :

$$\underbrace{(2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}}_{\text{Tilting}} + \underbrace{2\boldsymbol{\Omega}(\boldsymbol{\nabla}\cdot\boldsymbol{u})}_{\text{Stretching}} = \underbrace{-\frac{1}{\rho^2}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}_{\text{Baroclinic term}} \tag{3.10}$$

   If we write the equation along each axis (x,y and z), and retaining only the vertical component of the vorticity vector, we obtain :

$$f\frac{\partial u}{\partial z} = -\frac{1}{\rho^2}\left[\frac{\partial p}{\partial z}\frac{\partial \rho}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial \rho}{\partial z}\right] \tag{3.11a}$$

$$f\frac{\partial v}{\partial z} = +\frac{1}{\rho^2}\left[\frac{\partial p}{\partial z}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial z}\right] \tag{3.11b}$$

   (the z equation has been omitted since it is not immediately relevant here). We can show that in the rhs of (3.11a), the second terms vanish in pressure coordinates, and we find again the thermal wind balance, which is therefore a statement of vorticity conservation!

   In z coordinates, the argument is that we are comparing variations in $\bar{\rho}(z)$ to perturbations ρ', hence it can we shown that the first terms on the RHS are dominant (this is subtle).

[5] this requires the pressure field to be a C^2 function

4 Bridge over shallow water

4.1 Shallow water theory

Constant density implies that fields don't vary on the vertical : all the motion is due to **height gradients**, because $\frac{1}{\rho}\boldsymbol{\nabla} P = g\boldsymbol{\nabla} h$, where h is the thickness of the layer, generally split-up in $h = H + \eta$ (mean + perturbation). We can also include the effect of a variable bottom ($H(x, y)$) but we won't here. The rotating shallow water equations are:

$$\frac{du}{dt} - f\,v = -g\frac{\partial h}{\partial x} \tag{4.1a}$$

$$\frac{dv}{dt} + f\,u = -g\frac{\partial h}{\partial y} \tag{4.1b}$$

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) \simeq \frac{dh}{dt} + H\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \tag{4.1c}$$

The model is widely used to understand purely dynamical mechanism (no T and S here...) that can then be incorporated into layer models. It is in particular crucial to understand external/barotropic waves (topographic Rossby waves, gravity waves, etc...) and all of equatorial wave theory. It is more used in the ocean than the atmosphere, with the Gill model a notable exception.

4.2 Layered models

1. Two-layer model :

   Explain the relationship between interface displacements of a two layer stratified fluid and the surface elevation. What are the orders of magnitude of both parameters accross the Gulf Stream and along the Equatorial Pacific ?

   2 layer model : If the interface never outcrops, the lower layer (density ρ_2) must be at **rest**. Therefore pressure gradients in the layer must be null (otherwise there would be a non-zero velocity field to balance them).

$$h = \bar{h} - \frac{\rho}{\Delta\rho}\eta \tag{4.2a}$$

   or

$$\eta = (\bar{h} - h)\frac{\Delta\rho}{\rho} \tag{4.2b}$$

   Since $\frac{\Delta\rho}{\rho} \sim 2\times 10^{-3} \ll 1$, the interface displacement is always much greater than the surface perturbation: a 10 cm surface displacement implies a ~50m thermocline displacement in the Tropical Pacific. The 40 cm east-west climatological sea level difference must therefore correspond to a thermocline depth difference of about 200m, which is roughly correct. It is about twice as small in the Gulf Stream region, and much smaller in non dynamically-exciting regions.

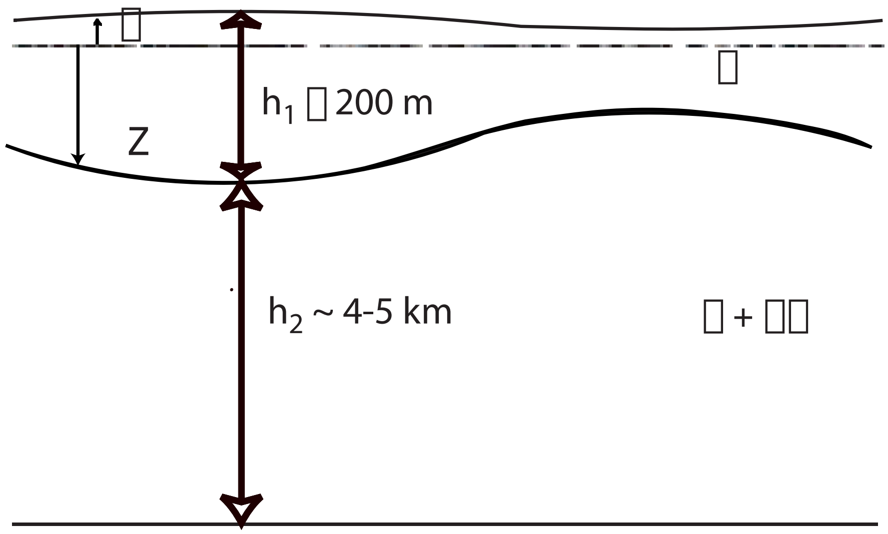

Figure 1: *Two layer model (flat bottom).* The surface instantaneous elevation is η and the depth of the interface is Z, whose average is $\bar{Z}$
.

The relationship between the thickness of layer 1 and the other variables is $h_1 = \eta - Z$. Thus, using (4.1a), we get the familiar relationship:

$$g\boldsymbol{\nabla}\eta = g\frac{\Delta\rho}{\rho}\boldsymbol{\nabla}h_1 = g'\boldsymbol{\nabla}h_1 \tag{4.3}$$

where $g' = g\frac{\Delta\rho}{\rho}$ is the **reduced gravity**. Most often, 2-layer models (e.g. the ocean component of the Zebiak-Cane model) work this way.

2. Relationship with normal modes In a N-layer system, you get N modes (one barotropic and $N-1$ baroclinic).

5 Quasi-geostrophic theory

As seen in section 2.2, for a judicious range of non-dimensional numbers, the velocity field can be considered almost geostrophic (within about 10%), the remainder being therefore a small correction, which we found was on the order of the Rossby number. However, because geostrophic equilibrium does not involve time derivatives, it does not allow any type of prediction, therefore all the interesting dynamics is embedded precisely in the "small correction" terms.
The idea is then to use the vorticity constraint on the flow as a predictive equation, the advection being done by the geostrophic part of the flow. Because this flow is strictly non-divergent on the f-plane, the only divergence must be **ageostrophic**: this is the only source of vertical motion (therefore rain, snow, etc...), and we need a way to estimate it. Since the vertical acceleration is so small in most situations, the hydrostatic balance is very robust, and does not give any information about

vertical motion. This information is obtained via the heat equation to eliminate w from the momentum equations. Here is a recapitulation of the notation used in the theory (from Lien Hua's excellent GFD class):

ATMOSPHERE		OCEAN
	Coordinates	
(x, y, p, t)		(x, y, z, t)
Geopotential Φ		Streamfunction $\psi = \frac{p}{\rho_\circ f_\circ}$
	Geostrophy	
$\boldsymbol{v}_g = \frac{1}{f_\circ}\boldsymbol{k} \times \boldsymbol{\nabla}\Phi$		$\boldsymbol{v}_g = \boldsymbol{k} \times \boldsymbol{\nabla}\psi$
	Hydrostatic equilibrium	
$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$		$p_z = -\rho g$
$T = -\frac{p}{R}\frac{\partial \Phi}{\partial p}$		$\rho = -\frac{\rho_\circ f_\circ}{g}\frac{\partial \psi}{\partial z}$
	Heat Conservation	
	$\frac{D_g}{Dt} = \partial_t + \boldsymbol{u}_g \cdot \boldsymbol{\nabla}$	
$\frac{D_g T}{Dt} - \left(\frac{\sigma p}{R}\right)\omega = 0$		$\frac{D_g \rho}{Dt} - \left(\frac{\rho_\circ N^2}{g}\right) w = 0$
$\sigma = -\frac{RT_\circ}{p\theta_\circ}\frac{d\theta_\circ}{dp} \quad ; \quad \omega = \frac{dp}{dt}$		$N^2 = -\frac{g}{\rho_\circ}\frac{d\bar{\rho}}{dz} \quad ; \quad w = \frac{dz}{dt}$
	Potential vorticity	
	$\frac{D_g}{Dt} q = 0$	
$q = f_\circ^{-1}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_\circ}{\sigma}\frac{\partial \Phi}{\partial p}\right)$		$q = \nabla^2\psi + f + \frac{\partial}{\partial z}\left(\frac{f_\circ^2}{N^2}\frac{\partial \psi}{\partial z}\right)$

The expression for the **quasi-geostrophic potential vorticity** is obtained as follows (ex, in the ocean) : Write the pressure gradient force as f times the geostrophic velocity, and the Lagrangian derivative as :

$$\frac{D_g}{Dt} = \partial_t + u_g \partial_x + v_g \partial_y \tag{5.1}$$

The frictionless, beta-plane momentum equations become :

$$\frac{D_g u}{Dt} - f(v - v_g) = 0 \tag{5.2}$$

$$\frac{D_g v}{Dt} + f(u - u_g) = 0 \tag{5.3}$$

This is crucial, because it says that it is the difference between the total velocity and the geostrophic velocity (i.e., the *ageostrophic* component) which rules the evolution of the total velocity field. Taking the Curl of these equations, we get the vorticity equation :

$$\frac{D_g}{Dt}(\zeta_g + f) = -f \boldsymbol{\nabla}_H \cdot \boldsymbol{u}_H = f\left(\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w)\right) \tag{5.4}$$

Using the heat equation (cf table), we can estimate w and give it in terms of the streamfunction. Using geostrophy, we also express the velocity field in these terms. Finally the equation reduces to a non-linear, but single, PDE for the streamfunction (or geopotential height), called the **quasi-geostrophic potential vorticity equation** :

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right)\left[\nabla^2 \psi + f + \frac{1}{\rho_s}\frac{\partial}{\partial z}\Big(\frac{\rho_s}{S(z)}\frac{\partial \psi}{\partial z}\Big)\right] = 0 \tag{5.5}$$

(S is the stratification parameter, or the inverse of the Burger number, roughly of order one in QG theory). This relation states that the quantity in brackets is conserved along streamlines (i.e., by the geostrophic flow) hence deserving the name "potential" vorticity. Because this equation involves time derivatives, it is said *prognostic* (unlike geostrophy, which is purely *diagnostic*), and can tell us about the future behavior of the system, at least from one day to the next (after that, neglected terms start messing up the prediction). It is the cornerstone of modern dynamic meteorology (and got also very successful in oceanography): give me the QGPV at time t, and I'll give you a pretty good guess for the state of the system at $t + \Delta t$ with minimal computational expense.
In fact the theory is amazingly successful for a vast range of scales (synoptic to planetary) outside the Tropics (this was investigated by Charney, 1963, who explained why it didn't work there), as long as $\epsilon < 10^{-1}$: the equation also works in a spherical geometry, given suitable definitions of the coordinates, and shows that the Earth's curvature matters mostly through the beta-effect, not so much the so-called curvature terms. It is used for weather prediction, explains the major instabilities, geostrophic adjustment, Rossby wave propagation, etc.. From a geographical point of view, it is hard to say where, exactly, it does break down, but a short-hand answer is "one equatorial Rossby radius away from the equator", i.e. about 15 degrees North or South. In practice, it is used in midlatitudes, and the corresponding dynamical paradigm in the Tropics is equatorial wave theory (attention, the latter does not allow weather forecasting, unlike QG theory).

6 Instability theory

> *The phenomenon of instability is the phenomenon of the preferential transfer of energy from the wave-free flow to the fluctuating flow*
> Joseph Pedlosky, Geophysical Fluid Dynamics, p490

From the dynamical constraints applied by potential vorticity conservation, it is possible to derive *necessary* conditions for the instability of a particular basic state described by the structure of the zonal wind $U_0(y, z)$. These are conditions that the large-scale flow must meet in order for certain criteria of instability to be valid (for example, that $\partial_t \eta^2 > 0$, where η is some measure of the meridional displacement associated with a perturbation). We may look for *normal mode solutions* of the form :

$$\phi(x, y, z, t) = \Re\{\ \Phi(y, z) e^{\imath k(x - ct)}\} \tag{6.1}$$

where $c = c_r + \imath c_i$. If solutions with $kc_i > 0$ are found, they will grow exponentially, which is typical of the synoptic scale disturbances we wish to explain. In addition, we look for the *most unstable* growth rate $\max(kc_i)$, since, if it exists, will give the preferred scale generated by the instability. The rationale for this procedure is that if conditions are favorable to the growth of this mode, then it will tap into the energy of the mean-flow *faster than any other mode*, and exhaust it before other modes had their say. This leaves behind a more stable mean flow and eddies with the characteristics of this fastest-growing mode. It is therefore a very scale-selective process (and the scale is, as usuals, the Rossby radius of deformation). But first, let's show how perturbations can grow. If we linearize around an x-independent basic state, the vorticity conservation writes:

$$\underbrace{\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{\rho_s}\frac{\partial}{\partial z}\frac{\rho_s}{S(z)}\frac{\partial\phi}{\partial z}\right)}_{\text{advection of anomalous PV by the mean flow}} + \underbrace{\frac{\partial\phi}{\partial x}\frac{\partial\Pi_0}{\partial y}}_{\text{advection of mean PV by perturbations}} = 0 \qquad (6.2)$$

where $\Pi_0 = f + \frac{\partial^2\Psi}{\partial y^2} + \frac{1}{\rho_s}\frac{\partial}{\partial z}\frac{\rho_s}{S(z)}\frac{\partial\Psi}{\partial z}$ is the potential vorticity of the mean flow. (quadratic terms have been sent to hell).

If (6.2) is multiplied by $\rho_s\phi$ and integrated over the volume of the flow, and neglecting friction, we obtain a tendency equation for the energy of the perturbation $E(\phi)$:

$$E(\phi) \equiv \int_0^{z_t}\int_{-1}^{+1} dydz\frac{\rho_s}{2}\left[\overline{u^2} + \overline{v^2} + \overline{\frac{1}{S}\theta^2}\right] \qquad \text{kinetic + potential} \qquad (6.3a)$$

$$\frac{\partial E(\phi)}{\partial t} = -\int_0^{z_t}\int_{-1}^{+1} dydz\rho_s\left[\underbrace{\overline{uv}\,\frac{\partial U_0}{\partial y}}_{\text{"Barotropic" momentum flux}} - \underbrace{\overline{v\theta}S^{-1}\frac{\partial U}{\partial z}}_{\text{"Baroclinic" heat flux}}\right] \qquad (6.3b)$$

Therefore, in the absence of friction, sources and sinks, and other junk, this wonderful equation states that **a perturbation of a geophysical flow can only exploit 2 sources of energy** : the momentum flux due to the meridional shear of the mean flow, or the heat flux associated with the vertical shear of the mean zonal flow (closely related to the meridional temperature gradient via the infamous thermal wind balance). The first mechanism is thus called **barotropic** instability (no dependence on z) and the second **baroclinic** instability, highly dependent on vertical shear.

6.1 Baroclinic instability

6.1.1 Basic mechanism : slanted convection

PICTURE from Pedlo p519
In a rotating and stratified system, isotherms can slope with respect to isobars, even at steady-state, because the thermal-wind is here to maintain that gradient. *Pedlosky* (1987) nicely defines an analog of the buoyancy restoring force in the case of isotherms that are sloping towards the pole :

$$F_R = \frac{g}{\theta_\circ}\frac{\partial\theta}{\partial z}\sin\alpha\left[\zeta - \eta\left(\frac{\partial z}{\partial y}\right)_\theta\right] \qquad (6.4)$$

Clearly if the slope is such that :

$$\boxed{0 < \tan\alpha < \left(\frac{\partial z}{\partial y}\right)_\theta}$$

then the restoring force is negative (i.e., not restoring anymore) and instability can ensue ; light fluid rises and cold fluid sinks, which releases the available potential energy of the basic state. This is the

fundamental reason for baroclinic instability, which is therefore a form of thermal convection along slanted isentropes, hence called **slanted convection**.
Here are the two most famous model of baroclinic instability : Eady's and Charney's. Naturally, both use quasi-geostrophic theory. Eady's model is outrageously simplified from a physical point of view, but that does make itmathematically tractable, whereas Charney's model makes more physical sense but is hardly solvable by paper and pen (unless you're Charney).

6.1.2 Eady's model

Simplifications :

- Linear shear $U(z) = \Lambda z \quad \therefore \quad \frac{\partial \Theta_\circ}{\partial y} = -\Lambda$
- $\beta = 0, \qquad S, \rho_s = cst$
- Hence, **there is no meridional gradient of ambient potential vorticity** (weirdissimo)

For simplicity let's take $\Lambda = 1$. Looking for normal-mode solutions, we must solve :

$$(z-c)\left\{S^{-1}\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial y^2} - k^2\phi\right\} = 0, \tag{6.5}$$

with boundary condition

$$-c\frac{\partial\phi}{\partial z} - \phi = 0 \qquad \text{at } z = 0, \tag{6.6}$$

$$(1-c)\frac{\partial\phi}{\partial z} - \phi = 0 \qquad \text{at } z = 1. \tag{6.7}$$

Moreover, $v = 0$ in $y = \pm 1 \Rightarrow \Phi \sim \cos(l_n y) \quad (l_n = (n+1/2)\pi, n \in \mathbb{N}^*)$. The problem becomes, for $\Phi(y,z) = A(z)\cos(l_n y)$:

$$(z-c)\left[\frac{d^2A}{dz^2} - \mu^2 A\right] = 0 \tag{6.8}$$

(and suitably adapted BC). $\mu = \sqrt{S(k^2 + {l_n}^2)}$ is the total wavenumber measured in units of the internal Rossby deformation radius. The solution is in terms of $\cosh\mu z$ and $\sinh\mu z$, and applying the BC requires that non-trivial solutions verify:

$$c^2 - c + \mu^{-1}\coth\mu - \mu^{-2} = 0 \tag{6.9}$$

this is a quadratic equation for c, as a function of μ only. Only if $\mu < \mu_c = 2.4$ can we obtain a solution with $c_i > 0$: 2 complex conjugates roots with $c_r = 0.5\ (\overline{U})$. The most unstable mode is the least wiggly in y ($n = 0$). For $S = 0.25$, we find its wavelength to be:

$$\lambda_* \approx 4L_D \tag{6.10}$$

thus the crest-to-trough distance (a quarter wavelength) is $L_D = 1000\ \mathrm{km}$ i.e. precisely the scale of weather systems. To good to be true!

Notes :

- $|\Phi(z)|$ has a maximum at $z = 1/2$, i.e. where $U_\circ = c_r$ (steering level, cf Charney's model): this is a critical level (a level where the waves extract energy from the mean flow), but it does not matter for the solution here.
- The perturbation heat flux is everywhere northward (average effect of weather systems on the mean flow in the northern Hemisphere).
- Major shortcomings :
  1. No meridional gradient of ambient potential vorticity ($\frac{\partial \Pi_\circ}{\partial y} \equiv 0$)
  2. The finite depth of the layer is a necessary condition for instability (otherwise $c_i \rightarrow 0$ as $z \rightarrow \infty$).

6.1.3 Charney's model

The most important additions to Eady's model are the consideration of an infinite, compressible atmosphere. Hence, the scale height is finite : $H = \left(\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z}\right)^{-1} < \infty$. Moreover, and this is the most interesting one, there is a meridional gradient of ambient potential vorticity. $\frac{\partial \Pi_\circ}{\partial y} \neq 0$. The curvature of the Earth is retained as well as the mean stratification, and we have :

$$\frac{\partial \Pi_\circ}{\partial y} = \beta + \frac{1}{SH} \tag{6.11}$$

Again we look for solutions $\phi = A(z)\cos(l_n y)e^{\imath k(x-ct)}$ to (6.2). We get :

$$(z - c)\left[\frac{d^2 A}{dz^2} - \frac{1}{H}\frac{\partial A}{\partial z} - \mu^2 A\right] + D\left(\frac{1}{h_*} + \frac{1}{H_*}\right) A = 0 \tag{6.12}$$

which should be compared to (6.8). The last term, which equals $\beta S + H^{-1}$, combines the THREE vertical scales of the problem :

1. D, the scale of the motion
2. h_*, such that
$$h_* = {\beta_\circ}^{-1}\frac{f_\circ^2}{N_s^2}\frac{\partial U_*}{\partial z_*}$$
3. H_*, the scale height.

A very novel feature of this model is that D is set by the motion itself via the instability process. The existence of this last term renders it impossible to escape some critical layers where $U(z) = z = c$. However, if we look at unstable modes ($c_i > 0$), these values of z are not in the real domain and we will not consider these singularities here (although they are very important on a mathematical point of view). In most cases, $D \sim h_*$ and the problem simplifies to :

$$(z-c)\left[\frac{d^2A}{dz^2}-\delta\frac{\partial A}{\partial z}-\mu^2 A\right]+(\delta+1)A=0 \tag{6.13}$$

where $\delta = h_*/H_*$. BC :

$$c\frac{dA}{dz}+A=0, \quad z=0 \tag{6.14}$$

$$\rho_s|A|^2 \in \mathbb{R}, \quad z\to\infty \tag{6.15}$$

Transform :

$$A=(z-c)\,e^{\nu z}F(z), \quad \nu=\frac{\delta}{2}-\left(\mu^2+\frac{\delta^2}{4}\right)^{1/2} \tag{6.16}$$

note that as the scale height goes to infinity, $\nu \to -\mu$, i.e. the vertical scale is proportional to the horizontal scale ! Thus we introduce the new vertical coordinate $\xi = (z-c)(\delta^2+4\mu^2)^{1/2}$, and we must solve :

$$\xi\frac{d^2F}{d\xi^2}+(2-\xi)\frac{dF}{d\xi}-(1-r)F=0, \qquad r=\frac{\delta+1}{(\delta^2+4\mu^2)^{1/2}} \tag{6.17}$$

The solution is in terms of confluent hypergeometric function $M(a,2,\xi)$ and $U(a,2,\xi)$ (Abramowitz and Stegun, 1964), with $a = 1-r$. Depending on the eigenvalues a, we obtain different vertical structures (in particular, whether or not $r \in \mathbb{N}^*$) since r gives the relationship between μ and δ. However, the family of curves indexed by r in the μ-δ plane are not marginality curves, and the flow is everywhere unstable except on lines corresponding to integral r.
Well, the general solution is devilishly complex, but in the incompressible case, the final result is extraordinarily close to Eady's answer. In fact, the most unstable growth rate is :

$$kc_i=\begin{cases}0.31\frac{f_\circ}{N}\frac{\partial U}{\partial z} & \text{(Charney)}\\ 0.29\frac{f_\circ}{N}\frac{\partial U}{\partial z} & \text{(Eady)}\end{cases} \tag{6.18}$$

the main difference is that in Charney's model, the heat transport is intensified near the surface, and the horizontal scale is a function of β :

$$k^{-1}=1.26\left(\frac{f_\circ}{\beta N}\right)\frac{\partial U}{\partial z} \tag{6.19}$$

whereas in Eady's case :

$$k^{-1}=0.8\,\frac{NH}{f} \tag{6.20}$$

Howeer, in Charney's case, there is no cutoff at high k's : instabilities can occur at all scales, but the maximum growth rate is obtained for k as in (6.19) Also, in Charney's model , the vertical scale depends on the relevant parameters of the problem (and is not prescribed):

$$\mu^{-1}=1.26\,\frac{f_\circ}{\beta}\left(\frac{\partial y}{\partial z}\right)_\theta \text{ (function of rotation and the slope of isentropes)} \tag{6.21}$$

6.1.4 What do these models tell us ?

In the pure baroclinic case, (6.3a) becomes :

$$\frac{\partial E(\phi)}{\partial t} \propto \int_0^{z_t} \int_{-1}^{+1} dydz \ \overline{v\theta}\frac{\partial U}{\partial z} \tag{6.22}$$

In most relevant cases, $\frac{\partial U}{\partial z}$ is positive (the zonal wind increases with height), therefore for disturbances to grow, we must have $\overline{v\theta} > 0$: **the perturbations must transport heat northward** (poleward, more generally). But :

$$\overline{v\theta} \propto \overline{\frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial z}} = -\left(\frac{dz}{dx}\right)_\phi \left(\frac{\partial \phi}{\partial z}\right)^2 \tag{6.23}$$

therefore $\overline{v\theta} > 0 \Longleftrightarrow$ "phaselines tilt westward with height". This is indeed what is systematically observed in weather systems and storms : the geopotential height trough at 500 mb is always west of the surface trough. Another way of justifying this is given in the review by Jessie Cherry on storms & hurricanes. One way to remember this: the perturbations are **leaning** on the mean flow (U being greater at height, and westerly in midlatitides). The situation is very similar for barotropic instability.

6.1.5 Necessary conditions : the Charney-Stern theorem

Gill (1982) goes through a nice idealized example very similar to the previous models, but where disturbances do not grow. The example shows that "the mere presence of available potential energy (APE)[6] is not sufficient to ensure instability since dynamical constraints may not allow the energy to be released". In Eady's model and Charney's model, appropriate boundary conditions allowed a westward phase tilt of the temperature field with height, thereby implying a "poleward" heat flux (and thus an APE release). But is that always possible? From the energy equations, it is possible to extract useful criteria from both the real and imaginary parts, which show that for c_i (unstable growth rate) to be positive, the following criterion must be met:

$$\text{Charney-Stern theorem} \quad \iint \frac{\partial \bar{q}}{\partial y}\frac{|\psi|^2}{|\bar{u}-c|^2} dydz + \int_{-L}^{+L} \Im \left[\frac{f_\circ^2}{N^2}\left(\frac{\partial \bar{\Theta}}{\partial y}\right)\frac{|\psi|^2}{|\bar{u}-c|^2}\right]_{z_\circ}^{z_1} dy = 0 \tag{6.24}$$

This implies that the 3 quantities :

$$\left(\frac{\partial \bar{q}}{\partial y}\right) \qquad -\left(\frac{\partial \bar{\Theta}}{\partial y}\right)_{\text{top}} \qquad \left(\frac{\partial \bar{\Theta}}{\partial y}\right)_{\text{bottom}} \tag{6.25}$$

must change sign over the domain of integration (they must include both positive and negative values). In Eady's model, $\frac{\partial \bar{q}}{\partial y} \equiv 0$ and we can show that the meridional gradients at the top and

[6] APE is defined as the difference between the potential energy and the minimum PE that the system can reach by an isentropic redistribution of mass. The idea is that not all of the PE (which is several orders or magnitude greater than KE at large scales) can be converted into motion, since the overwhelming fraction of the energy is stored in the mean stratification (stable). Therefore APE is really the "available" fraction of that energy that perturbations can tap into.

the bottom exactly cancel each other : the Charney-Stern criterion is always met. In Charney's model, the radiation condition at the top forces the corresponding term to vanish in (6.24). However, $\left(\frac{\partial\bar{\Theta}}{\partial y}\right)_{\text{bottom}}$ is critical: in summer it is weak, and the instability criterion is not met. In a winter situation it is large, and the criterion may be met. This explains why some disturbances can grow baroclinically unstable in winter and not in summer.

6.2 Barotropic instability

The purely barotropic case is one in which only the $\frac{\partial U}{\partial y}$ term contributes to $\partial_t E$ in (6.3a). The perturbations can grow from the kinetic energy of the mean current, provided a criterion akin to Charney-Stern is met. In that case it is the meridional shear of zonal velocity which matters, and the criterion implies that $\beta - \frac{\partial^2 U}{\partial y^2}$ must change sign in the domain. Barotropic instability is common in laboratory flows, and is not at all typical of geophysical flows (requiring neither rotation, stratification, nor thin shells. In fact, the β term is stabilizing, because it forces the curvature of U to exceed a threshold value). However, this instability tends to occur ubiquitously in the atmosphere and the ocean where $\frac{\partial U}{\partial y}$ is large ; this typically arises where $\frac{\partial U}{\partial z}$ is large, so in practice it tends to occur together with baroclinic instability (in clear application of Murphy's law). The result is a lot of eddies everywhere in the ocean and in the atmosphere... I believe barotropic instability is especially important in tropical oceans, where the shears are enormous between North and South equatorial currents and countercurrents. This gives rise to Tropical Instability Waves (TIW), of wavelength comparable to the equatorial Rossby radius (300 km), and which seems to affect the equatorial heat budget quite a bit.
This concludes this elementary overview of GFD. If not of this makes sense, read the following books!

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